ON KINEMATICALLY CONTROLLED MECHANICAL SYSTEMS

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"Kinematically controlled systems" are mechanical systems with constraints depending on parameters which can be varied while the system moves. A suitable program for the variation of these parameters depending on the current state of the system and on time, could impose on the motion some desired properties.

Kinematically controlled systems are closely connected with the servo-systems of Begen-Appell[1]. Appell's servo-systems are different from systems usually considered in mechanics. In our work we develop a new approach to such systems. They are treated from the beginning as controlled systems. We introduce the concept of a parametric constraint (Section 1) as the basic characterizing element of a kinematically controlled system. We show equations for the possible displacements of the system (Section 2), we analyze the correctness of the law for the control of the system (Section 3), we give the modification of the Gauss principle for kinematically controlled systems (Section 5). The indices used in this paper take on the following values

$$i = 1, 2,\ldots, 3n;\quad \rho, \pi = 1,2,\ldots, r;\quad \sigma,\tau = 1,2,\ldots, s;\quad \nu = 1,2,\ldots, k;\ \xi = 1, 2,\ldots, r+s$$

1. A system of n material points moves with respect to a Cartesian coordinate system. Let $m_1 = m_2 = m_3$, x_1, x_2, x_3 be the masses and the coordinates of the first point of the system, $m_4 = m_5 = m_6$, x_4, x_5, x_6 be the masses and the coordinates of the second point, and so on. Let some bodies without mass constrain the motion of the points of the system forming holonomic constraints which are given by Equations

$$f_\rho(x_1,\ldots, x_{3n},\quad t) = 0 \tag{1.1}$$

Such constraints can be realized if the bodies constraining the systems are freely absorbed by it, or if their motion (or deformation) is assigned initially. We shall allow however that among the bodies constraining the system there will be such whose motion (or deformation) depends on parameters which could suitably vary while the system moves. Such constraints will be called "constraints depending on parameters", or simply parametric constraints, and the parameters will be called the control parameters.

Let us consider, for example, a ring sliding freely on a rod which is hinged at one of its ends. If the conditions of the problem are such that we can vary the angle of inclination between the vertical and the rod while the ring is sliding, then the constraint on the ring will be parametric and the angle of inclination between the rod and the vertical will be the control parameter.

Let us denote by $p_1, \ldots, p_k$ the control parameters of he considered mechanical system and let Equations of its parametric constraint be

$$\varphi_\sigma(x_1, \ldots, x_{3n}, \ t, \ p_1, \ldots, \ p_k) = 0 \tag{1.2}$$

As in (1.1) Equations (1.2) are obtained from the conditions of preserving the contact between the points of the system and the constraining bodies. When $t, p_1, \ldots, p_k$ are fixed then Equations (1.2) and (1.1) express the geometry of the constraints of the system.

In the considered example the question of the parametric constraint is written in the form $x_2 = x_1 \tan\theta$, where θ is the angle of inclination between the rod and the vertical (the control parameter), x_1, x_2 are the coordinates of the ring with the origin of the coordinate system coinciding with the hinge, and the x_1-axis is along the vertical. The geometry of the constraint is a line passing through the origin.

Equations of the parametric constraints (1.2) contain the control parameters. Assigning to the parameters definite values which depend on the current state of the system and on time, we are controlling the system in its motion. In the given case the control will be kinematic, since it is realized through the constraints of the system. If control of the system is realized through forces, then it is a dynamic control. When the expression for the control parameters of a system contain its state parameters and the time, then it will be called "the law of control of the system". Let the law of control for our mechanical system be

$$p_\nu = p_\nu \ \ (t, x_1, \ldots, x_{3n}, \ \ x_1', \ldots, x_{3n}') \tag{1.3}$$

where $x_1', \ldots, x_{3n}'$ are the velocity components of the points of the system along the coordinate axes.

Parametric constraints are generalizations of the conventional, time depending constraints and reduce to them if the law of control depends only on time.

2. The constraints of the system are assumed to be ideal, that is such, that the work of the reactions of the constraints for all possible displacements equals identically zero. The possible displacements of a system will be understood as they are usually understood for the holonomic systems, meaning all possible infinitely small displacements of the system occuring in every considered instant of time and obeying the geometry of the constraints. Let us denote by $R_1, \ldots, R_{3n}$ the components of the reactions of the constraints along the coordinate axes, by $\delta x_1, \ldots, \delta x_{3n}$ the components of all the possible displacements of the system. Then condition for the constraints of the system to be ideal can be written as

$$\sum R_i \delta x_i = 0$$

This Equation should be valid for all possible displacements of the system. By Newton's law

$$R_i = m_i x_i'' - X_i$$

Here x_1'', x_2'', x_3'', X_1, X_2, X_3 are, respectively, the components of the real acceleration of the first point of the system and of the force acting on it along the coordinate axes; x_4'', x_5'', x_6'', X_4, X_5, X_6 are, respectively, the components of the real acceleration of the second point and of the force acting on it, and so on. Consequently

$$\sum (m_i x_i'' - X_i)\, \delta x_i = 0 \tag{2.1}$$

In this way we have obtained the fundamental equation of mechanics. In the considered case accelerations of all the points of the system are real and the equation satisfied for all possible displacements of the system. In this way we have established that the principle of D'Alembert-Lagrange is applicable to the controlled mechanical systems.

We shall derive now the equations of motion for the controlled system which we consider. Let us write first the equations for all the possible displacements of the system. By definition of the possible displacements these Equations have the usual form

$$\sum \frac{\partial f_\rho}{\partial x_i}\, \delta x_i = 0, \qquad \sum \frac{\partial \varphi_\sigma}{\partial x_i}\, \delta x_i = 0 \tag{2.2}$$

Let us mention, however, that the relations (2.2) were obtained by considering $t, u_1, \ldots, u_k$ as constants; in other words, the differentiation in (2.2) is carried out only with respect to the explicitly appearing coordinates.

Now, by the usual procedure, we derive from (2.1) and (2.2) Equations

$$m_i x_i'' = X_i + \sum \lambda_\rho \frac{\partial f_\rho}{\partial x_i} + \sum \mu_\sigma \frac{\partial \varphi_\sigma}{\partial x_i} \tag{2.3}$$

where λ_ρ and μ_σ are undetermined multipliers. Equations (2.3) together with constraint Equations (1.1) and (1.2) and control Equations (1.3) form a complete system of equations determining accelerations of the points of the system and the undetermined multipliers λ_ρ and μ_σ. Equations (2.3) are equations of motion of a kinematically controlled system in the form of Lagrange equations with multipliers.

For example, let us write the equations of motion of the ring in the previously mentioned example. Taking into consideration that the ring is subjected only to the parametric constraint

$$\varphi\ (x,\ y,\ \theta) = x_2 - x_1\ \mathrm{tg}\ \theta = 0 \tag{2.4}$$

we obtain from (2.3) desired Equations

$$mx_1'' = X_1 - \mu\ \mathrm{tg}\ \theta, \qquad x_2'' = X_2 + \mu$$

These equations have to be taken simultaneously with (2.4) and with the law of control $\theta = \theta\ (t,\ x_1,\ x_2,\ x_1',\ x_2')$.

Similarly to the systems with conventional constraints for which we can formulate theorems on the motion of the center of mass and on the angular momentum, we can formulate these theorems for systems with parametric constraints.

A. If among all the possible displacements of a system there is a translatory displacement when a system is moving as a rigid body along any fixed direction, then the center of mass of the system moves along this direction, thus the system can be replaced by a point containing the mass of the whole system, and the force acting on this point equals the sum of all active forces acting on the system.

B. If among all the possible displacements of a system there is a rotation of a system when a system rotates as a rigid body about a fixed axis, then the time derivative of the angular momentum of the system about this axis equals the moment about this axis of all the active forces acting on the system.

We prove these theorems using the relations (2.1) and the proof does not differ from the one for systems with conventional constraints.

The theorem on kinetic energy does not apply to the systems with parametric constraints. A controlled system moves, in general, while the values of the control parameters vary. Under these conditions the real displacements of the system are not among the possible displacements, and the formal condition for the kinetic energy theorem is not satisfied. The kinetic energy theorem does not apply to the systems with parametric constraints because the kinetic energy of such a system varies not only on account of the work done by active forces acting on the system but also on account of the actions forced by the control of the system.

3. By a "correct law" we shall call such a law of control of a system, which will make the motion possible and unique. The requirement of correctness is not trivial. Let us show some simple examples of incorrect laws of control.

Let us assume that a point mass is subjected to the parametric constraints

$$x_1 + x_2 = p_1, \qquad x_1 - x_2 = p_2 \tag{3.1}$$

where p_1 and p_2 are the parameters of control. Let us prescribe control Equations

$$p_1 = x_3, \qquad p_2 = -2x_2 + x_3$$

in one case and Equations

$$p_1 = x_3, \qquad p_2 = -2x_2 + x_3 + 1$$

in another case. Equations (3.1) are independent. The system of equations obtained from the above equations by eleminating the control parameters is in the first case dependent and in the second case incompatible.

A more refined phenomenon can be illustrated on the example of a ring slipping freely on a smooth rod.

Let us prescribe Equations controlling the rod

$$\theta = \sin^{-1} \frac{x_2}{a}$$

where a is a constant. From above Equation we shall find the control parameter in the constraint equation. We obtain

$$x_1^2 + x_2^2 = a^2$$

The ring ought to move on the circumference of radius a. But such a motion of a ring is, in general, impossible. In this way the prescribed control equation for the system is forcing a condition on the motion of a ring, which cannot be satisfied.

Let us find for the control equation the condition of correctness. It consists obviously in this, that the system of Equations (2.3),(1.1),(1.2) and (1.3) should determine uniquely accelerations of the system for every permissible state of the system. Equations (2.3) give explicit expressions for accelerations of the system through the forces and undetermined multipliers. Therefore, the condition for correctness of the control equations reduces to the condition of a unique determination of the undetermined multipliers λ_ρ and μ_σ. Using Equations (1.3) we shall eliminate the control parameters from Equations (1.1) and (1.2). By taking the time derivatives of Equations (1.1) and (1.2) we obtain Equations

$$\sum A_{\xi i} x_i'' + A_\xi = 0 \tag{3.2}$$

for the kinematically permissible accelerations of the particles of the system. The number of these equations equals the number of Equations (1.1) and (1.2), therefore it equals the number of multipliers λ_ρ and μ_σ. Substituting the expressions for $x_1'', \ldots, x_{3n}''$ from Equations (2.3) into (3.2), we obtain the following system which determines the undetermined multipliers λ_ρ and μ_σ

$$\sum a_{\xi\rho}\lambda_\rho + \sum b_{\xi\sigma}\mu_\sigma + c_\xi = 0 \tag{3.3}$$

where

$$a_{\xi\rho} = \sum \frac{1}{m_i} A_{\xi i} \frac{\partial f_\rho}{\partial x_i}, \qquad b_{\xi\sigma} = \sum \frac{1}{m_i} A_{\xi i} \frac{\partial \varphi_\sigma}{\partial x_i}, \qquad c_\xi = \sum \frac{1}{m_i} A_{\xi i} X_i + A_\xi \quad (3.4)$$

The system (3.3) has a unique solution, which means that the control equation is correct, if the determinant of the system

$$\Delta = \begin{vmatrix} a_{11} \ . \ . \ . \ b_{11} \ . \ . \ . \\ a_{21} \ . \ . \ . \ b_{21} \ . \ . \ . \\ . \ . \ . \ . \ . \ . \ . \ . \ . \end{vmatrix} \qquad (3.5)$$

does not vanish.

In order that the control equation be regular it is necessary, in particular, that the system of Equations (3.2) be linearly independent.

Indeed, if Equations (3.2) are dependent, then there exist multipliers $\varkappa_\xi$, not all of them zero, such that Equations

$$\sum \varkappa_\xi A_{\xi i} = 0$$

are valid.

Multiplying above Equations, respectively, by $(1 / m_i)\, \partial f_\rho / \partial x_i$ and adding them we obtain, using the notation in (3.4)

$$\sum \varkappa_\xi a_{\xi\rho} = 0$$

Similarly, we find

$$\sum \varkappa_\xi b_{\xi\sigma} = 0$$

In this way the elements of the determinant Δ are linearly dependent. This means that $\Delta = 0$ and consequently the control equation is incorrect. That is what we wanted to show.

4. In order to exhibit the inner contents of the requirements of correctness, in the cosidered metric space of the variables x_i, we shall prescribe the metrics of this space in the form

$$ds^2 = \sum m_i dx_i^2$$

and integrate the motion of the system as a motion of a point in this space.

The accelerations $x_1'', \ldots, x_{3n}''$ of the points of our system determine the acceleration of the integrated point. We shall call it the acceleration of the system.

We shall resolve every kinematically permissible acceleration of the system into two components, one which is tangent and one which is normal to the geometry of the constraint of the system.

We have

$$x_i'' = u_i + v_i \tag{4.1}$$

The set of all the possible displacements of the system forms a linear space, tangent to the geometry of the constraints of the system. Therefore the components u_i of the tangential component of the acceleration of the system should satisfy the equation of the possible displacements

$$\sum \frac{\partial f_\rho}{\partial x_i} u_i = 0, \qquad \sum \frac{\partial \varphi_\sigma}{\partial x_i} u_i = 0 \tag{4.2}$$

The components v_i of the normal component of the acceleration, for all possible displacements, should satisfy Equation

$$\sum m_i v_i \delta x_i = 0 \tag{4.3}$$

expressing**the condition of orthogonality of the vectors $v_1, \ldots, v_{3n}$ and $\delta x_1, \ldots, \delta x_{3n}$ in our metric space.

From the condition (4.3) and from Equations (2.1), for all possible displacements, we find for the normal component of the acceleration of the system

$$m_i v_i = \sum \lambda_\rho^* \frac{\partial f_\rho}{\partial x_i} + \sum \mu_\sigma^* \frac{\partial \varphi_\sigma}{\partial x_i}$$

where λ_ρ^* and μ_σ^* are arbitrary multipliers. Taking into account these expressions, Equations (4.1) become

$$x_i'' = u_i + \sum \lambda_\rho^* \frac{1}{m_i} \frac{\partial f_\rho}{\partial x_i} + \sum \mu_\sigma^* \frac{1}{m_i} \frac{\partial \psi_\sigma}{\partial x_i} \tag{4.4}$$

L e m m a 1 . Every kinematically permissible acceleration of the system has a unique component tangent to the geometry of the constraints of the system.

It is obvious that to prove this lemma it is sufficient to show that when $x_1'', \ldots, x_{3n}''$ are prescribed, then we can select uniquely the multipliers λ_ρ^* and μ_σ^* so that $u_1, \ldots u_{3n}$, determined by Equations (4.4), would satisfy the relations (4.2).

For this purpose we substitute Equations (4.4) into (4.2). Using the notation

$$p_{\rho\varkappa} = \sum \frac{1}{m_i} \frac{\partial f_\rho}{\partial x_i} \frac{\partial f_\varkappa}{\partial x_i}, \qquad q_{\rho\tau} = \sum \frac{1}{m_i} \frac{\partial f_\sigma}{\partial x_i} \frac{\partial \varphi_\tau}{\partial x_i}$$

$$r_{\sigma\varkappa} = \sum \frac{1}{m_i} \frac{\partial \varphi_\sigma}{\partial x_i} \frac{\partial f_\varkappa}{\partial x_i}, \qquad s_{\sigma\tau} = \sum \frac{1}{m_i} \frac{\partial \varphi_\sigma}{\partial x_i} \frac{\partial \varphi_\tau}{\partial x_i}$$

** The general geometric theory of the metric spaces can be found in the book [2].

we obtain the Equations

$$\sum p_{\rho\pi}\lambda_\pi{}^* + \sum q_{\rho\tau}\mu_\tau{}^* = \sum \frac{\partial f_\rho}{\partial x_i} x_i'', \qquad \sum r_{\sigma\pi}\lambda_\pi{}^* + \sum s_{\sigma\tau}\mu_\tau{}^* = \sum \frac{\partial \varphi_\sigma}{\partial x_i} x_i'' \quad (4.5)$$

which should be satisfied by the undetermined multipliers $\lambda_\pi{}^*$ and $\mu_\tau{}^*$, so that $u_1, \ldots, u_{3n}$ would possess the property required by the lemma.

We shall prove that the determinant of this system does not vanish. Let us assume that it vanishes. Then we can find $\varkappa_\rho$ and ν_σ not all zero, such that following Equations will be satisfied

$$\sum \varkappa_\rho p_{\rho\pi} + \sum \nu_\sigma r_{\sigma\pi} = 0, \qquad \sum \varkappa_\rho q_{\rho\tau} + \sum \nu_\sigma s_{\sigma\tau} = 0$$

Eliminating in them $p_{\rho\pi}$, $r_{\sigma\pi}$, $q_{\rho\tau}$, and $s_{\sigma\tau}$ and introducing the notation

$$\omega_i = \sum \varkappa_\rho \frac{\partial f_\rho}{\partial x_i} \frac{1}{m_i} + \sum \nu_\sigma \frac{\partial \varphi_\sigma}{\partial x_i} \frac{1}{m_i} \quad (4.6)$$

we shall reduce these Equations to the form

$$\sum \frac{\partial f_\pi}{\partial x_i} \omega_i = 0, \qquad \sum \frac{\partial \varphi_\tau}{\partial x_i} \omega_i = 0 \quad (4.7)$$

Multiplying Equations (4.6), respectively, by $m_i \omega_i$ and adding them we obtain

$$\sum m_i \omega_i{}^2 = \sum \varkappa_\rho \frac{\partial f_\rho}{\partial x_i} \omega_i + \sum \nu_\sigma \frac{\partial \varphi_\sigma}{\partial x_i} \omega_i$$

By (4.7) the right member of this Equation, equals identically zero. Consequently, all $\omega_i = 0$. But then by (4.6) comes out that Equations of constraints (1.1) and (1.2) depend on each other, which contradicts the hypothesis.

In this way we have shown that the determinant of the system (4.5) does not vanish, which means that the undetermined multipliers $\lambda_\pi{}^*$ and $\mu_\tau{}^*$ are unique. The Lemma is proved.

L e m m a 2 . If the control equations of the system are correct, then the tangential component of the kinematically permissible acceleration of the system can be arbitrarily prescribed; the kinematically permissible acceleration is uniquely determined by its tangential component.

It is obvious that we prove the Lemma if we show that for any system $u_1, \ldots, u_{3n}$, the multipliers $\lambda_\rho{}^*$ and $\mu_\sigma{}^*$ can be uniquely selected in such a way that Equations (4.4) would determine kinematically permissible acceleration of the system. For this purpose we substitute Equations (4.4) in the relations (3.2) and we obtain Equations for the kinematically permissible acceleration, which, using the notation (3.4), are

$$\sum a_{\xi\rho}\lambda_\rho{}^* + \sum b_{\xi\sigma}\mu_\sigma{}^* + \sum A_{\xi i} u_i + A_\xi = 0 \quad (4.8)$$

and the multipliers λ_ρ^* and μ_σ^* must satisfy these Equations to atisfy the conditions of the Lemma. Since the control equations have to be correct the determinant of this system (the determinant (3.5)) does not vanish and consequently the undetermined multipliers λ_ρ^* and μ_σ^* are unique. The Lemma is proved.

The two Lemmas permit the formulation of the following Theorem:

If the constraints (1.1) and (1.2) are independent, and the control equations of the system are correct, then we have a one to one correspondence between the kinematically permissible accelerations of the system and their tangential components. Besides, the set of the tangential components of the accelerations is unbounded (with the exception, of course, of the determining relations (4.2)).

5. The Gauss principle as applied to a system with conventional constraints formulates the extremal property of the real acceleration, selecting it among the kinematically permissible accelerations of the system.

The proper Gauss principle is not applicable to a system with the parametric constraints, however a variation of it, formulated below, does apply.

Among the kinematically permissible accelerations of a system the real one possesses a tangential component which makes the function

$$\sum \frac{m_i}{2}\left(u_i - \frac{X_i}{m_i}\right)^2 \tag{5.1}$$

a minimum.

To prove this we shall substitute the expansion (4.1) in the fundamental equation of mechanics (2.1). Taking into account condition (4.3) we obtain

$$\sum (m_i u_i - X_i)\, \delta x_i = 0 \tag{5.2}$$

The quantities $u_1, \ldots, u_{3n}$ represent the tangential component of the real acceleration of the system. We shall select arbitrarily a kinematically permissible acceleration of the system. Let us denote by $w_1, \ldots, w_{3n}$ its tangential component. Since the two tangential components (u and w) satisfy (4.2), their difference must also satisfy (4.2). Consequently, the differences $u_i - w_i$ determine a vector of a displacement of the system. This means that Equation (5.2) can be rewritten as

$$\sum (m_i u_i - X_i)\,(u_i - w_i) = 0$$

Using the identities

$$\left(u_i - \frac{X_i}{m_i}\right)(u_i - w_i) = \frac{1}{2}\left[\left(u_i - \frac{X_i}{m_i}\right)^2 - \left(w_i - \frac{X_i}{m_i}\right)^2 + (u_i - w_i)^2\right]$$

we find

$$\sum \frac{m_i}{2}\left(u_i - \frac{X_i}{m_i}\right)^2 - \sum \frac{m_i}{2}\left(w_i - \frac{X_i}{m_i}\right)^2 + \sum \frac{m_i}{2}(u_i - w_i)^2 = 0$$

From which we obtain

$$\sum \frac{m_i}{2}\left(u_i - \frac{X_i}{m_i}\right) \leqslant \sum \frac{m_i}{2}\left(w_i - \frac{X_i}{m_i}\right)^2$$

and this is what we wanted.

In this way we have proved that the minimum of the function (5.1) corresponds to the tangential component of the real acceleration of the system. The same applies to any function

$$\sum \frac{m_i}{2}(u_i - u_i^*)^2 \qquad (5.3)$$

where $u_1^*, \ldots, u_{3n}^*$ is the tangential component of the real acceleration of the system, which has been relieved of some of the constraints.

Indeed, for a system which is relieved of some of the constraints, the relation (5.2) is written as

$$\sum (m_i u_i^* - X_i)\,\delta^* x_i = 0$$

where $(\delta^* x_1, \ldots, \delta^* x_{3n})$ is the possible displacement of the relieved system. The possible displacements of the initial system are a subject of the possible displacements of the relieved system. Consequently, last Equation can be written as

$$\sum (m_i u_i^* - X_i)\,\delta x_i = 0$$

Substracting it from Equation (5.2), we obtain

$$\sum m_i (u_i - u_i^*)\,\delta x_i = 0$$

Replacing (5.2) by the above relation and repeating word by word the previously carried considerations, we obtain the previously obtained result with the difference, that the function (5.1) is replaced by (5.3).

The function (5.1) is a special case of the function (5.3) when the system is relieved of all constraints.

We shall show that the modified Gauss principle applied to the case of conventional constraints reduces to the proper Gauss principle.

We are going to show first, that in the systems with conventional constraints the normal component $(v_1, \ldots, v_{3n})$ of the kinematically permissible accelerations of the system is independent of its tangential component.

Indeed, in the preceding article we had for the component $(v_1, \ldots, v_{3n})$ the following relations:

$$m_i v_i = \sum \lambda_\rho^* \frac{\partial f_\rho}{\partial x_i} + \sum \mu_\sigma^* \frac{\partial \varphi_\sigma}{\partial x_i}$$

If $(u_1, \ldots, u_{3n})$ is the tangential component of the kinematically permissible acceleration of the system, then the undetermined multipliers λ_ρ^* and μ_σ^* should be calculated from the relations (4.8). It is seen from (3.2) that in a system with conventional constraints we can set

$$\| A_{\xi i} \| = \left\| \begin{matrix} \partial f_\rho \,/\, \partial x_i \\ \partial \varphi_\sigma \,/\, \partial x_i \end{matrix} \right\|$$

On the other hand u_i satisfies the relations (4.2). Consequently, for the conventional systems

$$\Sigma A_{\xi i} u_i = 0$$

and the quantities u_i drop out from (4.8). In this way the undetermined multipliers λ_ρ^* and μ_σ^*, and consequently the quantities v_i, are independent of the kinematically permissible tangential component of acceleration. This is what we required.

From these considerations follows, that the normal component of the acceleration of the system is only a function of the state of the system and all the kinematically permissible accelerations of the system in its prescribed state differ only in their tangential components.

Let $u_1 + v_1, \ldots, u_{3n} + v_{3n}$ be the real acceleration of the system. Then its arbitrary kinematically permissible acceleration, as we have shown, can be written as $w_1 + v_1, \ldots, w_{3n} + v_{3n}$, where $w_1, \ldots, w_{3n}$ is its tangential component.

By using Gauss's principle and by (4.1) we can write the function which is minimized as

$$z_w = \sum \frac{m_i}{2} \left(w_i + v_i - \frac{X_i}{m_i} \right)^2$$

Let z_u be the value of our function corresponding to the real acceleration of the system.

We shall consider the difference

$$z_u - z_w = \sum \frac{m_i}{2} \left(u_i + v_i - \frac{X_i}{m_i} \right)^2 - \sum \frac{m_i}{2} \left(w_i + v_i - \frac{X_i}{m_i} \right)^2$$

By expanding it we find

$$z_u - z_w = \sum \frac{m_i}{2} \left(u_i - \frac{X_i}{m_i} \right)^2 - \sum \frac{m_i}{2} \left(w_i - \frac{X_i}{2} \right)^2 + \sum m_i (u_i - w_i) v_i$$

The differences $u_i - w_i$, as we have mentioned before, determine the vector of a possible displacement of the system. Consequently, by (4.3) the last sum in the right-hand term of the above Equation vanishes and

$$z_u - z_w = \sum \frac{m_i}{2}\left(u_i - \frac{X_i}{m_i}\right)^2 - \sum \frac{m_i}{2}\left(w_i - \frac{X_i}{m_i}\right)^2$$

This last equation proves what we wanted.

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